Algebraic Topology : midterm

1. Suppose a simplicial complex structure on a closed surface of Euler characteristic χ has v vertices, e edges, and f faces, which are triangles. Show that $e = 3f/2, f = 2(v - \chi)$, $e = 3(v-\chi)$, and $e \leq v(v-1)/2$. Deduce that $6(v-\chi) \leq v^2 - v$. For the torus, conclude that $v \geq 7, f \geq 14$, and $e \geq 21$.

   Remark: In fact, for the torus, the minimum values $(v, e, f) = (7, 14, 21)$ can be realized by a simplicial structure on the torus. You are not asked to show this.

2. The degree of a homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ can be defined as the degree of the extension of f to a homeomorphism of the one-point compactification S^n. Using this notion, show that $\mathbb{R}^n$ is not homeomorphic to a product $X \times X$ when n is odd.

   Hint: Assuming $\mathbb{R}^n = X \times X$, consider the homeomorphism f of $\mathbb{R}^n \times \mathbb{R}^n = X \times X \times X \times X$ that cyclically permutes the factors, $f(x_1, x_2, x_3, x_4) = (x_2, x_3, x_4, x_1)$.

3. Let $\mathbb{R}^n$ be the union of two open subsets U and V. Show the following.

   (a) If U and V are path connected, then $U \cap V$ is path connected.

   (b) If any two of the sets $\pi_0(U), \pi_0(V)$ and $\pi_0(U \cap V)$ are finite, then the third is also finite. Moreover, we have

   $$|\pi_0(U \cap V)| + |\pi_0(U \cup V)| = |\pi_0(U)| + |\pi_0(V)|$$

   where $|X|$ means the cardinality of the set X.

   (c) Suppose $x, y \in U \cap V$ can be connected by a path in U and by a path V, then x and y can be connected by a path in $U \cap V$.

   Hint: use homology theory. On the other hand, can you prove these statements directly from the definition of path components without the use of homology?

4. Let $(X_1, X_2, \cdots, X_n)$ be an open covering of X and $(Y_1, Y_2, \cdots Y_n)$ be an open covering of Y. Suppose $f\colon X \to Y$ is a continuous map such that $f(X_i) \subset Y_i$, and moreover the restriction

   $$f\colon \cap_{i \in A} X_i \to \cap_{i \in A} Y_i$$

   induces an isomorphism on homology for each subset $A \subset \{1, 2, \cdots, n\}$. Show that $f_*\colon H_n(X) \to H_n(Y)$ is an isomorphism for all n.

5. The Borsuk-Ulam theorem states that: any odd continuous map $f\colon S^n \to S^n$ must have odd degree. Here we say f is odd if $f(-x) = -f(x)$ for all $x \in S^n$. Let us assume the Borsuk-Ulam theorem throughout this exercise.

   (1) Prove that there does not exist an odd continuous map $g\colon S^n \to S^{n-1}$. Here again g is odd means that $g(-x) = -g(x)$.

   (2) Prove that for every continuous map $h\colon S^n \to \mathbb{R}^n$, there exists a point $x \in S^n$ with $h(x) = h(-x)$. This is often illustrated by saying that at any given moment, there are always two antipodal places on earth with equal temperatures and equal air pressures. (Hint: use part (1))

(3) Prove that if $S^n = F_1 \cup F_2 \cup \cdots \cup F_{n+1}$ where each F_j is a closed subset of S^n, then at least one of the sets F_j contains a pair of antipodal points. (Hint: consider distance functions to F_j, and use part (2))

(4) In previous parts, we have seen that (1) $\implies$ (2) $\implies$ (3). In fact, one can also show (3) $\implies$ (1) as follows. Observe that there exist closed subsets $F_1, \cdots, F_{n+1}$ of S^{n-1} such that $S^{n-1} = F_1 \cup F_2 \cdots \cup F_{n+1}$ and no F_i contains a pair of antipodal points. (For example, consider the standard n-dimensional simplex Δ^n, which is inscribed in a sphere S^{n-1}. Now take radial projection the boundary $\partial\Delta^n$ of Δ^n to this sphere. Note that $\partial\Delta^n$ consists of $(n+1)$ faces. Let F_i be the image of a corresponding face.) Use this observation to show that (3) $\implies$ (1).

(5) Here is a slightly different but equivalent version of part (3). Prove that if $S^n = A_1 \cup A_2 \cup \cdots \cup A_{n+1}$ where each A_j is either open or closed in S^n, then at least one of the sets A_j contains a pair of antipodal points. (Hint: use part (3))